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Testing the impartiality of surveys to measure differential risk perception

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ABSTRACT

The paper addresses the validation of opinion measurement by survey, specifically how the "readings" or views are consolidated for policy use into a single figure held to be representative of the population as a whole. While the sample mean might seem the obvious choice, a number of influential safety studies have employed a more complicated metric, the Valuation Index, as discussed within, to assess whether more should be spent to protect against some hazards than others. The question arises as to whether the Valuation Index treats the views of the different people in the survey sample impartially or not.

The Valuation Index possesses the property of reciprocality, in the sense that the Index of the reciprocal is equal to the reciprocal of the Index. Jensen's inequality reveals that this reciprocality comes at the price of an inconsistent treatment of the individual's view whenever there is any difference of opinion in the sample. The Valuation Index is found to be non-unique, in that a Second Valuation Index can be constructed using equivalent arguments. Thus if the Valuation Index is valid, then so must be the Second Valuation Index, but the latter returns a different value, implying a contradiction that can be resolved only by regarding neither as valid.

Crucially, both the Valuation Index and the Second Valuation Index fail the test of structural view independence. Both indices are biased and thus neither is a valid measure for consolidating human opinions.

The finding has particular importance for the UK because opinion surveys interpreted using the Valuation Index have been influential in the changes to safety policy brought in for UK rail transport, where the amount recommended to avert death in multiple fatality accidents since 2003 has been reduced to about a third of what it used to be.

1. Introduction

Measurement of economic parameters has enormous importance for the way in which we live our lives, particularly when they are used by governments and others to set policy. The onus on the person or persons carrying out such measurements to ensure the fairness of the measurement is correspondingly great. The burden of this duty will not be especially irksome when the measurement is of value in a free market, where the measurement task reduces simply to finding the price on which buyer and seller agree. However, there are and always will be instances where no obvious market exists, particularly in the case of public goods such as clean air or, the continued survival of a rare species of plant or animal, where it is still desirable to know what value people place on the amenity so that a cost-benefit analysis may be carried out.

"Revealed preferences" may enable an inferential measurement of the value of the non-market good to be made. Such methods conform to John Locke's precept: "I have always thought the actions of men the best interpreters

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of their thoughts", with the good's value determined through observing what quantifiable benefit has been given up in order to secure a certain quantity of that good or a proxy, or else, if the good is undesirable, what quantifiable burden has been taken on to avoid it. Clearly, the revealed preference technique requires a model of the system, and, as a minimum, that model needs to be transparent so that it can be examined in full detail by a person possessed of the necessary mathematical skills.

Attempts may be made to measure the value of a non-market good using opinion polling or market-survey techniques. These "stated preference" methods are sometimes referred to as "contingent valuation" [11] since the estimates obtained are conditional or "contingent" on the features of the scenario presented to the respondents taking part in the survey [8]. The surveyor needs to ensure as far as possible that each opinion elicited is both true, in the sense that it represents the view of the respondent and not, for example, what he thinks he ought to say, and stable, in the sense that it will represent the respondent's feelings on the subject for a reasonable amount of time into the future. Once the views of the respondents have been gathered, their interpretation becomes a key issue, since opinion polls and market surveys are based on the notion that an opinion that is in some sense characteristic of an entire population can be measured fairly accurately from the statements made by people in a random sample drawn from that population.

Deriving from the success of measurement in the physical sciences, a sound philosophical and theoretical basis has been developed for the general process of measurement [14]. But as noted by Boumans [6], economic systems may be characterised formally as "soft", since they satisfy two out of the three Finkelstein criteria for being regarded as such: (i) they involve human action, perception, feeling, decisions and the like and (ii) they usually have significant size and complexity, making experimental determination of relations between system elements impractical in many cases. It can be anticipated, therefore, that the soft measurement task of measuring the value of a non-market good may bring difficulties that are generally greater than those associated with many physical measurements.

Finkelstein [15] states that "measurement owes its power, primarily, to the objectivity of its description". Lack of objectivity may arise from the features of the model necessary to interpret the raw measurement data and provide information on the property actually of interest, of which the measurand is normally only a manifestation. Indeed Finkelstein [15] observes that the observer/analyst in social sciences has been argued not to be objective, but to operate "on the basis of ideologically motivated theories". In a similar vein, measurement theorists Mari et al. [20] regard objectivity and inter-subjective testability as the critical features for the reliability and dependability of measurement. It is not surprising, therefore, that Mari and Ugazio [21] should see the validation of soft measurements as a topic of increasing importance, and hold that "the meaningfulness … of the obtained results depends on interpretive models, that become pivotal for the validation of measurements and results". Indeed, given the clear potential for subjectivity and the widely held suspicions thereof, it is clearly important to open up the soft measurement process to examination and test, including the interpretation and processing of measurement data.

The personal valuations given in an opinion survey constitute the readings in what is an exercise in soft measurement. The subsequent processing of these readings makes up a crucial component of the measurement task, analogous to the filtering of the signal in a physical measurement system, for example by Kalman filtering. This paper addresses the process by which the views measured in an opinion survey are filtered to give one consolidated view that is then considered representative of the population as a whole. In particular, the paper will investigate the statistic employed in the consolidation of views in a number of influential surveys used in the UK to set expenditure priorities on systems to protect against fatal accidents.

It has been shown [26] that when the consolidation method to be applied to human views can be represented as a sequential process of transformation, averaging and back-transformation, of all the possible increasing and differentiable transformations, only linear functions preserve the necessary impartiality. This implies that the arithmetic or sample mean is valid as an unbiased measure of human opinion, but that, for example, both the geometric mean and the root mean square are invalid.

However, a different statistic has been used in preference to the sample mean in a number of influential studies carried out in the UK to assess the amount of money that should be spent to avert death from accidents. These studies include that commissioned in 1998 jointly by the UK Health and Safety Executive, the UK Department of the Environment, Transport and the Regions, the UK Home Office and the UK Treasury [3,4], the 2000 study commissioned by the UK Health and Safety Executive [7] (both reported in [10]) and the 2008 study by the UK Rail Safety and Standards Board [12], further reported in Covey et al. [13]. This statistic was called "relativity" in Chilton et al. [10], "sample valuation ratio" in Covey et al. [12] and "valuation ratio" in Covey et al. [13]. For clarity and ease of discussion the statistic will be given a single name in this paper, the "Valuation Index".

The Valuation Index was used in the Chilton article (*op. cit.*) to conclude that:

> "while people's priorities are indeed sensitive to the combined influence of the number of deaths, the psychological characteristics of hazards and social amplification effects following a major accident, in practice (at least using our particular elicitation methodology) it is the number of deaths which would appear to dominate the quantitative judgements people give. That is, it would seem that maximising the number of deaths prevented is of primary importance to many people."

Indeed, the House of Lords Select Committee on Economic Affairs [16] made specific reference to Chilton et al. [10] in its Fifth Report, and noted that.

> "since 2003 the rail industry and the Rail Safety and Standards Board (RSSB) in particular have now abandoned the use of the two distinct VPFs [values of a prevented fatality] and have instead elected to apply a

common baseline VPF equal to the Department for Transport roads figure. For example, in its recent document, Valuing Safety, the RSSB notes that:

> 'The term 'gross disproportion' was used by some people in the past to describe the concept of using a higher VPF for multi-fatality accidents. We can see no justification, either in the Edwards judgement or in the balancing approach of risk against sacrifice, for this interpretation'."

The effect of RSSB's decision in 2003 to abandon the use of two different VPFs, one for single fatality accidents and a larger one for multi-fatality accidents, had the practical effect of reducing by a factor of about three the amount that rail operators were obliged to spend to avert an accident likely to result in multiple fatalities, as explained by Bearfield [5]:

> "Following a catastrophic train accident at Clapham Junction in 1988 a disproportion factor of 2.8 between costs and risks was adopted by the British Railways Board (BRB) for risk associated with multi-fatality train accidents. These accidents are associated with high levels of 'societal concern'. This figure was derived from research studies BRB had commissioned. Up until 2003/2004 stakeholders within the railway industry applied an approach that was consistent with this thinking using a multiplier of three. For other types of risk no multiplier was applied. After this time, on the advice of the Department for Transport, the industry stopped using this multiplier."

Jack [17] gave the clarification that the Office of Rail Regulation no longer required the duty holders in the UK railway industry to consider any potential socio-political response to a multi-fatality incident, which was taken to be a matter for Government and the regulators.

Given the apparent influence of studies using the Valuation Index on the views of the UK Government, the UK's Rail Safety and Standards Board and the UK rail regulator, and the resulting reduced requirement for safety spending on the part of the UK rail industry, it is clearly desirable to understand the properties of the Valuation Index fully, and to find out if it treats the views of the different people consulted in the survey impartially or not.

To these ends, a general mathematical formulation will be made of the Valuation Index, and its properties will be brought out. Its value relative to the sample mean will be examined, and it will be tested for impartiality using the concept of structural view independence [26]. The statistic's property of reciprocality, considered desirable by its proponents, will then be examined in the light of Jensen's inequality, and the consequential side-effects brought out. A discussion of the statistic's overall properties will then be given, following which conclusions will be drawn.

2. Mathematical description of scenario-dependent death matching

The method given in Chilton et al. [10] may be described and modelled as follows. It is desired to investigate the views of respondent k to two sets of circumstances and manners of dying, generalised as scenario A and scenario B. Imagine that individual k would be prepared to authorise a sum of money, V_{Ak}, to avert one death in the base scenario A and a possibly different sum of money, V_{Bk}, to avert one death in an alternative scenario B that is under investigation. Now suppose that data are available on a pairwise comparison between options A and B as given in Table 1, where respondent k deems the number of deaths,N_{Ak}, in scenario A to represent as undesirable an outcome as N_{Bk} deaths in scenario B. Since the individual regards the two outcomes as equally bad, it is reasonable to assume that he would be prepared to authorise equal expenditure to avert these deaths. Hence

$$N_{Ak}V_{Ak} = N_{Bk}V_{Bk} \tag{1}$$

The ratio, R_{BAk}, of the second amount, V_{Bk}, to the first, V_{Ak}, is then the "valuation ratio", given by:

$$R_{BAk} = \frac{V_{Bk}}{V_{Ak}} = \frac{N_{Ak}}{N_{Bk}} \quad k = 1, 2, \ldots, n \tag{2}$$

so that the valuation ratio is calculated relative to base scenario A. It may be seen that there has been no need to assign a figure to the "Value of a Prevented Fatality" (VPF) for any of the respondents. (In reality, we never prevent a fatality, but simply restore life expectancy to its value prior to exposure to the risk in question. Thus the acronym, VPF, ought really to be written VTPF, the value of a temporarily prevented fatality. See, for example, [23,25,24,22,27,28].)

It is now possible to estimate without bias the expected valuation ratio, $E(R_{BA})$, for the population as a whole from the sample mean:

$$E(R_{BA}) \approx \overline{R_{BA}} = \frac{1}{n}\sum_{k=1}^{n} R_{BAk} \tag{3}$$

where $\overline{R_{BA}}$ is the sample mean. We may note in passing that the weighting given to each person's valuation ratio is the same: the weighting factor is $1/n$ in each case. Moreover, an unbiased estimate for the variance of the population, $var(R_{BA})$, is then found easily from the sample data by:

$$var(R_{BA}) \approx \frac{1}{n-1}\sum_{k=1}^{n}\left(R_{BAk} - \overline{R_{BA}}\right)^2 \tag{4}$$

However Chilton et al. [10] suggested that the approach embodied in Eqs. (3) and (4) above of finding "the

Table 1
Number of deaths in scenarios A and B, $\boldsymbol{N}_{Ak}$ and $\boldsymbol{N}_{Bk}$, deemed to be equivalently bad.

Respondent number	Scenario A	Scenario B
1	N_{A1}	N_{B1}
2	N_{A2}	N_{B2}
$\vdots$	$\vdots$	$\vdots$
j	N_{Aj}	N_{Bj}
$\vdots$	$\vdots$	$\vdots$
k	N_{Ak}	N_{Bk}
$\vdots$	$\vdots$	$\vdots$
n	N_{An}	N_{Bn}

arithmetic mean of individual relativity ratios", one previously recommended by Jones-Lee and Loomes [19], was now "superseded". They recommended instead a new technique for analysing paired data of the form shown in Table 1, resulting in a new statistic, the Valuation Index.

Implicit in the Valuation Index is the assignment of a monetary value to the VPF, and in the first instance we may suppose that each respondent sets his own valuation, VPF_k, in whatever currency he wishes. VPF_k is the base unit or *numéraire* for respondent k. Two convenient possibilities present themselves at this point. In Convention 1, respondent, k, takes the amount he would authorise to avert death under the scenario he fears more as his reference point, and wants less spent on averting death in the scenario he fears less. In this case 1 VPF_k is the maximum that respondent k is prepared to have spent to prevent one fatality under either scenario. In Convention 2, it is supposed that respondent, k, takes as his reference point the amount he would sanction to avert death under the scenario he fears less, and wants more to be spent on averting death in the scenario he fears more. Now 1 VPF_k will be the minimum that respondent k is prepared to authorise to prevent one fatality under either scenario. The UK rail industry was effectively using Convention 2 between 1988 and 2003, when it had a baseline VPF, which it would increase by a factor of 2.8 to give a special VPF for multi-fatality accidents. By contrast, Chilton et al. [10] adopted Convention 1 as the basis for their Valuation Index.

To develop the mathematical model under Convention 1, let us first assume that death in scenario A is feared by respondent k as much or more than death in scenario B. Put simply, scenario A is more frightening to respondent k than scenario B, or at least as frightening. In such a case, he will make the number of scenario A deaths smaller than or equal to the number of deaths in scenario B, $N_{Ak} \leqslant N_{Bk}$, so that, by Eq. (2), his valuation ratio will be less than or equal to unity:

$$R_{BAk} \leqslant 1 \tag{5}$$

Each of those N_{Ak} more fearsome deaths will be valued at 1 VPF_k, so that

$$V_{Ak} = 1 \quad \text{for } R_{BAk} \leqslant 1 \tag{6}$$

while the VPF in scenario B will be given by applying Eq. (2):

$$V_{Bk} = R_{BAk} V_{Ak} = R_{BAk} \quad \text{for } R_{BAk} \leqslant 1 \tag{7}$$

where Eq. (6) has been used in the second step.

Now let us consider the alternative possibility that death in scenario A is feared less than death in scenario B by respondent k. In such a case, he will make the number of scenario A deaths greater than the number of deaths in scenario B, $N_{Ak} > N_{Bk}$, so that, by Eq. (2) the valuation ratio will be greater than unity:

$$R_{BAk} > 1 \tag{8}$$

By Convention 1, each of those N_{Bk} deaths in scenario B will be valued at 1 VPF_k, so that now

$$V_{Bk} = 1 \quad \text{for } R_{BAk} > 1 \tag{9}$$

while, using Eq. (9):

$$V_{Ak} = \frac{V_{Bk}}{R_{BAk}} = R_{BAk}^{-1} \quad \text{for } R_{BAk} > 1 \tag{10}$$

where the second step has made use of Eq. (9).

Eqs. (6) and (10) may be combined to give the more compact description:

$$V_{Ak} = \min\left(1, R_{BAk}^{-1}\right) \tag{11}$$

while Eqs. (7) and (9) may be combined into

$$V_{Bk} = \min(1, R_{BAk}) \tag{12}$$

If Convention 2 is adopted, on the other hand, then it can be shown by an analogous procedure that the VPF values for scenarios A and B for respondent k are the different values:

$$V_{Ak} = \max\left(1, R_{BAk}^{-1}\right) \tag{13}$$

$$V_{Bk} = \max(1, R_{BAk}) \tag{14}$$

It is important to realise that, while the VPFs under scenarios A and B, V_{Ak} and V_{Bk} respectively, are convention-dependent, the valuation ratios, R_{BAk}, $k = 1, 2, \ldots, n$, are not (since R_{BAk} depends directly on survey data: $R_{BAk} = N_{Ak}/N_{Bk}$).

Thus, the valuation ratio for respondent k may be recovered from Eqs. (11) and (12), valid under Convention 1, by:

$$R_{BAk} = \frac{V_{Bk}}{V_{Ak}} = \frac{\min(1, R_{BAk})}{\min\left(1, R_{BAk}^{-1}\right)} = \begin{cases} \frac{R_{BAk}}{1} = R_{BAk} & \text{for } R_{BAk} \leqslant 1 \\ \frac{1}{R_{BAk}^{-1}} = R_{BAk} & \text{for } R_{BAk} > 1 \end{cases} \tag{15}$$

Similarly the valuation ratio for respondent k may be recovered once again from the VPFs pertaining under Convention 2, Eqs. (13) and (14) by:

$$R_{BAk} = \frac{V_{Bk}}{V_{Ak}} = \frac{\max(1, R_{BAk})}{\max\left(1, R_{BAk}^{-1}\right)} = \begin{cases} \frac{1}{R_{BAk}^{-1}} = R_{BAk} & \text{for } R_{BAk} \leqslant 1 \\ \frac{R_{BAk}}{1} = R_{BAk} & \text{for } R_{BAk} > 1 \end{cases} \tag{16}$$

The fact that the valuation ratio, R_{BAk}, for respondent k can be recovered in both cases (Eqs. (15) and (16)) gives a further demonstration that it does not matter which Convention respondent k adopts in order to assign monetary value: both are equally valid as a valuation convention. Moreover, respondent j may adopt Convention 1 and respondent k may adopt Convention 2 and it will still be possible to calculate the valuation ratios, R_{BAj} and R_{BAk}. Thus the sample mean, which depends linearly on R_{BAk}, $k = 1, 2, \ldots, n$, and only on those values, will be unaffected by different choices of valuation convention amongst the respondents.

3. The properties of the Valuation Index, $I(\cdot)$

Let us now consider the Valuation Index, $I(R_{BA})$, which is found as the ratio of means:

$$I(R_{BA}) = \frac{\overline{V}_B}{\overline{V}_A} = \frac{\sum_{k=1}^{n} V_{Bk}}{\sum_{k=1}^{n} V_{Ak}} \tag{17}$$

where

$$\overline{V}_A = \frac{1}{n}\sum_{k=1}^{n} V_{Ak} \tag{18}$$

while

$$\overline{V}_B = \frac{1}{n}\sum_{k=1}^{n} V_{Bk} \tag{19}$$

The possibility for different respondents to choose a different valuation convention now disappears, as Chilton et al. [10] assume that Convention 1 will be adopted by all respondents:

> "Clearly, on the usual interpretation of individual matching responses, these ratios give each person's value for the prevention of a fatality in any particular context relative to his/her value for the prevention of a fatality in his/her higher-ranked context."

A ratio of means will satisfy the requirement for reciprocality, in the sense that the index of the reciprocal is equal to the reciprocal of the index, as desired by the authors of the Valuation Index as their "Condition 1" and as shown below:

$$I\left(\frac{1}{R_{BA}}\right) = I(R_{AB}) = \frac{\overline{V}_A}{\overline{V}_B} = \left(\frac{\overline{V}_B}{\overline{V}_A}\right)^{-1} = \frac{1}{I(R_{BA})} \tag{20}$$

The structure of the Valuation Index will now be examined further. Adopting Convention 1 and thus using Eqs. (11) and (12) in Eq. (17) gives:

$$I(R_{BA}) = \frac{\sum_{k=1}^{n} \min(1, R_{BAk})}{\sum_{k=1}^{n} \min\left(1, R_{BAk}^{-1}\right)} \tag{21}$$

But

$$\begin{aligned} \min(1, R_{BAk}) &= R_{BAk} \min\left(\frac{1}{R_{BAk}}, 1\right) \\ &= \min\left(1, R_{BAk}^{-1}\right) R_{BAk} \end{aligned} \tag{22}$$

Hence Eq. (21) may be re-expressed as:

$$I(R_{BA}) = \sum_{k=1}^{n} \pi_k R_{BAk} \tag{23}$$

where π_k is the weight put on individual k's opinion:

$$\pi_k = \frac{\min\left(1, R_{BAk}^{-1}\right)}{\sum_{m=1}^{n} \min\left(1, R_{BAm}^{-1}\right)} \tag{24}$$

If all the respondents fear scenario B less than scenario A or fear the two scenarios equally, then $V_{Ak} \geqslant V_{Bk}$ and so, from Eq. (2), each respondent will have a valuation ratio less than or equal to unity:

$$R_{BAk} \leqslant 1 \quad \text{for all } k \tag{25}$$

In this restricted case, when nobody finds death in scenario B more fearsome, the weighting on each person's view is the same:

$$\pi_k = \frac{1}{n} \quad \text{for all } k \tag{26}$$

in which case Eq. (21) returns the sample mean:

$$I(R_{BA}) = \frac{1}{n}\sum_{m=1}^{n} R_{BAm} = \overline{R}_{BA} \tag{27}$$

But the situation becomes rather different when anyone in the sample fears scenario B more than scenario A. Expanding Eq. (24) for the exclusive and exhaustive cases that the valuation ratio is either less than or equal to unity or greater than unity, $R_{BAk} \leqslant 1$ and $R_{BAk} > 1$, the weighting applied to the opinion of respondent k is:

$$\pi_k = \begin{cases} \dfrac{1}{\sum_{m=1}^{n} \min\left(1, R_{BAm}^{-1}\right)} & \text{for } R_{BAk} \leqslant 1 \\ \dfrac{1}{R_{BAk} \sum_{m=1}^{n} \min\left(1, R_{BAm}^{-1}\right)} & \text{for } R_{BAk} > 1 \end{cases} \tag{28}$$

It is clear from Eq. (28) that the views of all respondents who fear death under scenario B less than under scenario A and hence consider that $R_{BAk} \leqslant 1$ will be given a high and equal weighting. Meanwhile those respondents who fear death in scenario B more than death under scenario A, who will have a valuation ratio, $R_{BAk} > 1$, will find their views given a lower weighting. The presence of the valuation ratio, R_{BAk}, in the denominator when $R_{BAk} > 1$ in Eq. (28), second line, means that the more that a respondent fears death in scenario B, the more his view will be downgraded.

A simple numerical example illustrates the point. Assume 40 people have given their opinions of how much to spend on preventing scenario B and how much on preventing scenario A in terms of the number of deaths under each scenario that they would consider equivalent. For simplicity, the number of deaths under scenario B has been assumed kept constant at 10 for all respondents, so that they will have varied the equivalent number of deaths under scenario A. Suppose respondent 1 considered 1 death under scenario A to be equivalent to 10 deaths under scenario B, respondent 2 considered 2 deaths under scenario A to be equivalent, and so on, until respondent 40 considered 40 deaths under scenario A to be equivalent to 10 deaths under scenario B.

Fig. 1 shows the weighting, π_k, given to the view of respondent k against the number of deaths in scenario A he judged to be equivalent to 10 deaths under scenario B. It is clear that the Valuation Index is biased systematically

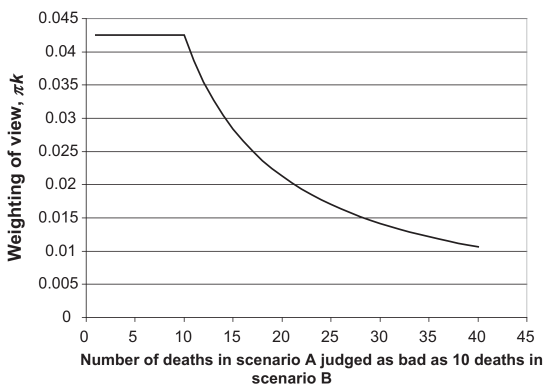

Fig. 1. Weighting of view, π_k, versus the number of deaths in scenario A judged to be as bad as 10 deaths in scenario B, example of Section 3.

against the views of those who fear death in scenario B more than death in scenario A.

4. The low value of the Valuation Index relative to the sample mean

The Valuation Index in the example just given was found to be 1.51, less than the sample mean, which was 2.05. In fact, the Valuation Index can be demonstrated to be less than or equal to the sample mean on all occasions.

Appendix A compares the figures returned by the Valuation Index and by the sample mean. It shows that the Valuation Index is equal to the sample mean in two circumstances only, both of which place severe constraints on the possible views of the individual:

(i) when all views are identical – the case of a degenerate probability distribution likely to occur only when a required response is dictated to the respondents, and
(ii) when no valuation ratio, R_{BAk}, is more than unity, implying that there is not even one respondent who fears scenario B more than scenario A.

In all other cases the Valuation Index will be less than the sample mean. Thus if there is a spread of opinions, where some people fear scenario A more than scenario B while others have a greater fear of scenario B, the Valuation Index of B relative to A must be less than the sample mean. In fact, it is necessary for only one respondent to fear scenario B more than scenario A for the Valuation Index, $I(R_{BA})$, to fall below the sample mean.

5. Using structural view independence to test the Valuation Index for impartiality

At this point it is appropriate to introduce the concept of structural view independence, a new and recently published criterion [26] for judging if the consolidation of human valuations into a single figure is objective. That paper found the sample mean to be the "gold standard" against which other means, such as the geometric mean or root-mean-square, should be judged. Central measures other than the sample mean can open the analyst to the charge of bias, but the sample mean protects the analyst from any suspicion of bias.

Defining a view as a person's judgement of the value of a continuous numerical parameter, a consolidated statistic is structurally view dependent when the process of consolidation involves using one or more of the views in the sample to filter an individual's view [26]. The Valuation Index, $I(R_{BA})$, will be shown to fail the test for structural view independence that is necessary for impartiality.

Consider the sensitivity of the Valuation Index to the views of individual respondents. The relevant sensitivity functions, $\partial I/\partial R_{BAk}$, are derived in Appendix B as equations (B.10) and (B.15). The form of the sensitivity function depends on whether or not the kth valuation ratio, R_{BAk}, is above or below unity:

$$\frac{\partial I}{\partial R_{BAk}} = \frac{1}{\sum_{m=1}^{n} \min\left(1, R_{BAm}^{-1}\right)} \qquad R_{BAk} \leqslant 1 \tag{29}$$

$$\frac{\partial I}{\partial R_{BAk}} = \frac{\sum_{m=1}^{n} R_{BAm} \min\left(1, R_{BAm}^{-1}\right)}{R_{BAk}^{2}\left(\sum_{m=1}^{n} \min\left(1, R_{BAm}^{-1}\right)\right)^{2}} \qquad R_{BAk} > 1 \tag{30}$$

It is clear from Eqs. (29) and (30) that the sensitivity function will depend on all the valuation ratios, R_{BAk}, $k = 1, 2, \ldots, n$, with the exception of the case where $R_{BAk} \leqslant 1$ for all k. In this case Eq. (29) is applicable, and reduces to

$$\frac{\partial I}{\partial R_{BAk}} = \frac{1}{n} \quad \text{when } R_{BAk} \leqslant 1 \quad \text{for all } k \tag{31}$$

Eq. (31) is, of course, consistent with Eq. (26), and leads to the sample mean of Eq. (27).

Eq. (30) demonstrates that the sensitivity will decrease as the inverse square of the valuation ratio when the valuation ratio is greater than unity, $R_{BAi} > 1$. Thus the Valuation Index is structurally view dependent, giving a high weight to the views of certain individuals in the sample while according others a lower influence, based purely on the size of their views. In fact, it exhibits a systematic bias towards low values.

The effect of this systematic bias will be illustrated by a numerical example, where the development of the Valuation Index is traced over time. Suppose that an organisation is spending the same amount of money to avert a death in scenario B as in scenario A but wants to know whether the public wants more money spent against B. It decides to consult an analyst, who uses a focus group to provide data that he interprets using the Valuation Index.

To keep the arithmetic clear, and for the purposes of illustration, it will be assumed that the focus group consists of just 7 people, who spend a working day considering the question of how many deaths in scenarios A and B each of them regards as equally bad. Their Table 1 is assumed to evolve, as will their valuation ratios, r_{BAk}, $k = 1, 2 \ldots, 7$, of desirable spending to avert a death under scenario B compared with the sum they would like to see spent to avert a death under scenario A. (The opinions of the members of the focus group are here regarded as realised values.)

It is assumed that each person starts the day with the view that death is equally to be feared in either scenario, so that $r_{BAk} = 1.0$ for all k. However, their views start to diverge as their judgements mature over the day. Persons 1–7 move towards final valuations, r_{BAkf}, of 3.0, 2.5, 2.0, 0.75, 0.65, 0.55 and 0.4, where the additional subscript, f, indicates "final". For simplicity and the purposes of display, the evolution of each of these views is assumed to take the form of an exponential lag with a time constant of 80 min, so that they have reached a steady state by the end of the day, six time constants or 8 h later. Appendix E outlines the mathematical model for how the respondent's views and hence the Valuation Index, I, change in the focus group during the day.

Fig. 2 shows the evolution over the day of the views of the seven people, and, in addition, the sample mean, $\overline{r_{BA}}$ and the Valuation Index, $I(r_{BA})$. Fig. 3 shows how the sensitivity functions, $\partial I/\partial R_{BAk}$, alter over time as the views change, while Table 2 summarises the starting and ending

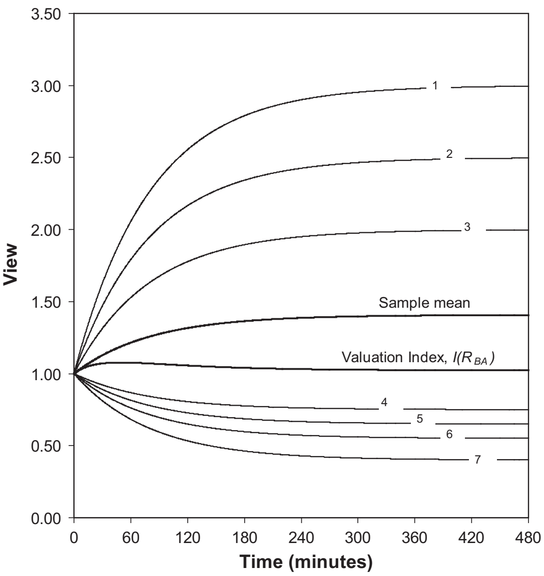

Fig. 2. Evolution of views in the numerical example of Section 5, value ratios for death in scenario B with death in scenario A as base.

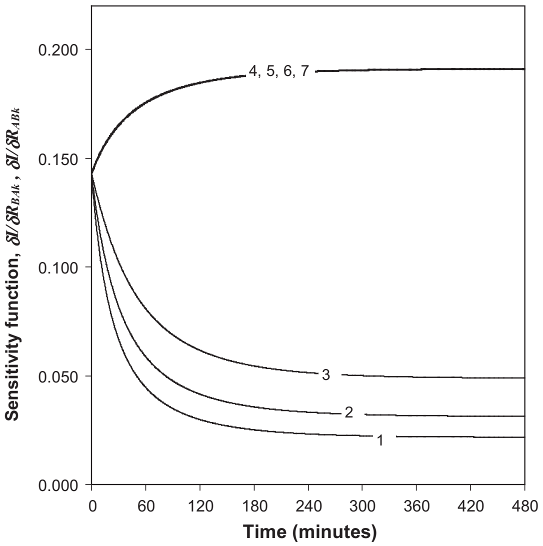

Fig. 3. Sensitivity functions for the 7 people in the numerical example of Section 5 when the Valuation Index is for death in scenario B with death in scenario A as base, $I(R_{BA})$.

values for: the views of the seven people, the sensitivity functions, the sample mean and the Valuation Index.

The Valuation Index in the steady state is $I(r_{BA}) = 1.022$, significantly less than the sample mean, $\overline{r_{BA}} = 1.407$. Thus while the sample mean would suggest that spending on averting each death under scenario B should be increased by 40%, the Valuation Index suggests that people are essentially indifferent, and that no further expenditure is necessary. Thus the organisation will save itself money if it takes advice based on the Valuation Index.

The fact that the Valuation Index always lies below the sample mean may be explained by reference to the sensitivity functions shown in Fig. 3. The sensitivity of the Valuation Index to each view is seen to depend strongly on the size of the various views. So while all sensitivities are the same at the beginning when everyone holds the same view, their magnitudes diverge as people's views evolve. The 4 views placing $r_{BA} \leqslant 1$ are always given the greatest influence: the sensitivities for Persons 4, 5, 6 and 7 are equal highest at all times, steadying out at 0.191. By contrast, those views placing $r_{BAk} > 1$ attract relatively little influence, and Person 1, who fears scenario B most, has his view downgraded most: the sensitivity is just 0.022 at the end of the day, down by an order of magnitude on the influence accorded those whose valuation ratios were less than or equal to unity.

But the numbers contained in the equivalent of Table 1 may be used to investigate not only whether the public wants more spent on averting a death in scenario B than in base scenario A, but also whether the public wants more spent on averting death in scenario A than in scenario B, since $r_{ABk} = r_{BAk}^{-1} = n_{Bk}/n_{Ak}$. Scenario B becomes now the base scenario. Fig. 4 shows the evolution of the valuation ratios, r_{ABk}, $k = 1, 2, \ldots, 7$, as well as the Valuation Index and the sample mean. Table 3 shows the starting and ending values of the views of the seven people, the sensitivity functions, the sample mean and the Valuation Index when scenario B is the base scenario. Once again the final value of Valuation Index is significantly less at $I(r_{AB}) = 0.978$ than the final value of the sample mean, $\overline{r_{AB}} = 1.203$. Thus while the sample mean would suggest that spending to avert a death under scenario A should be higher, the Valuation Index suggests that people are essentially indifferent, and that no further expenditure is necessary. Once again, if the organisation takes advice based on the Valuation Index, it will save itself money. But the Valuation Index has allowed this apparent economy only by discriminating strongly against the opinions of some people in the survey.

Fig. 5 shows the close-of-day sensitivity functions for each respondent, both $\partial I/\partial R_{BAk}$, when scenario A is the base scenario and $\partial I/\partial R_{ABk}$, when scenario B is the base scenario. The two sensitivities for the same person are different by up to an order of magnitude at the same time. Thus at the end of the day, Person 1 has a sensitivity of 0.187 when $I(r_{AB})$ is being calculated, but, simultaneously, a sensitivity of 0.022 when $I(r_{BA})$ is being calculated. Corresponding simultaneous figures for Person 7 are 0.029 and 0.191.

These biases and contradictions emerge as the price paid for the reciprocality displayed by the Valuation Index. To suggest that, because society might more spent to avert death in scenario A it automatically must want less spent to avert death in scenario B, as indicated by the Valuation Index, is to ascribe a certainty of view to society that is not possessed in a democracy. Reciprocality of view is not to be expected in an open society, where views are not imposed. And if views were to be imposed, what would be the point of an opinion survey?

So how do we account for the fact that the sample means are not reciprocal, not even close in this case, with $\overline{r_{BA}} = 1.407$ and $\overline{r_{AB}} = 1.203$ in the steady state? Is it not contradictory that the same focus group is urging 40% more to be spent on averting a death in scenario B than in scenario A, while at the same time, asking that 20% more

Table 2
Scenario A is the base scenario. Start and end of the day: valuation ratios and sensitivity functions at the for the 7 respondents in the focus group, sample mean, $\overline{r_{BA}}$, and Valuation Index, $I(r_{BA})$.

Respondent number, k	Start of day		End of day	
	Valuation ratio, r_{BAk}, for respondent, k	Sensitivity function $\frac{\partial I}{\partial R_{BAk}}$	Valuation ratio, r_{BAk}, for respondent, k	Sensitivity function $\frac{\partial I}{\partial R_{BAk}}$
1	1	0.1429	3.00	0.0218
2	1	0.1429	2.50	0.0313
3	1	0.1429	2.00	0.0490
4	1	0.1429	0.75	0.1910
5	1	0.1429	0.65	0.1910
6	1	0.1429	0.55	0.1910
7	1	0.1429	0.40	0.1910
Sample mean, $\overline{r_{BA}}$	1	–	1.41	–
Valuation Index, $I(r_{BA})$	1	–	1.02	–

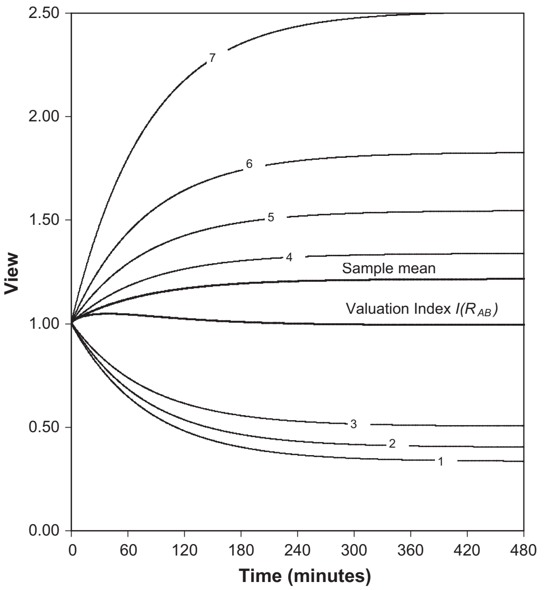

Fig. 4. Evolution of views in the numerical example of Section 5, value ratios for death in scenario A with death in scenario B as base, $I(R_{AB})$.

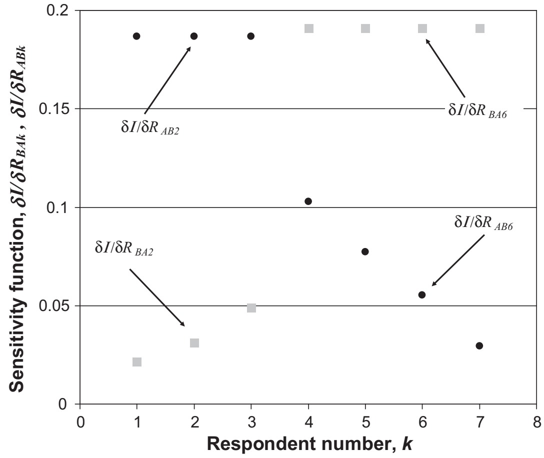

Fig. 5. Sensitivity functions of persons 1 and 7 at the end of the day for the numerical example of Section 5. In the one case the Valuation Index is for death in scenario B with death in scenario A as base, $I(R_{BA})$, while in the other the Valuation Index is for death in scenario A with death in scenario B as base, $I(R_{AB}) = I\left(R_{BA}^{-1}\right)$.

be spent to prevent a death in scenario A than in scenario B? In fact, what is being signalled is the strong divergence of opinion in the focus group. If such a wide split occurred in practice, surely it is this that should be explored first as a source of valuable information.

For example, is there a commonality, in the sense that each person wants the current spend against death in the less fearsome scenario to be maintained but extra to be

Table 3
Scenario B is the base scenario. Start and end of the day: valuation ratios and sensitivity functions at the for the 7 respondents in the focus group, sample mean, $\overline{r_{AB}}$, and Valuation Index, $I(r_{AB})$.

Respondent number, k	Start of day		End of day	
	Valuation ratio, r_{BAk}, for respondent, k	Sensitivity function $\frac{\partial I}{\partial R_{ABk}}$	Valuation ratio, r_{BAk}, for respondent, k	Sensitivity function $\frac{\partial I}{\partial R_{ABk}}$
1	1	0.1429	0.33	0.1868
2	1	0.1429	0.40	0.1868
3	1	0.1429	0.50	0.1868
4	1	0.1429	1.33	0.1030
5	1	0.1429	1.54	0.0774
6	1	0.1429	1.82	0.0554
7	1	0.1429	2.50	0.0293
Sample mean, $\overline{r_{AB}}$	1	–	1.20	–
Valuation Index, $I(r_{AB})$	1	–	0.98	–

spent to prevent death in his more frightening scenario? Does this mean that the amounts that members of the focus group think are being spent at present to avert a death in both scenarios are too low? If the expenditure on preventing a death in both scenarios were raised, would the disparity between opinions reduce? Do new arguments need to be presented if the opinions of both factions in the group are to be drawn closer?

These are open questions. It may be remarked that it is exceedingly difficult to please all people on an issue of importance to them, a proposition that will come as no surprise to politicians in a democracy. The non-reciprocality of the sample mean, the arithmetic average of the valuation ratios, would appear to reflect common democratic experience here.

6. The reciprocality of the Valuation Index

It is shown in Appendix A that the Valuation Index, $I(R_{BA})$, may be regarded as the expectation of the valuation ratio, R_{BA}, under the condition that the probability of selecting the view of individual is π_k:

$$I(R_{BA}) = \underset{\pi_k}{E^s}(R_{BA}) \tag{32}$$

Here the expectation operator, E, has been given the superscript, s, to signify that the population of interest in this case is the sample population and also marked with the discrete probability distribution, π_k, $k = 1, 2, \ldots, n$. By Jensen's inequality [18,9]

$$\underset{\pi_k}{E^s}\left(\frac{1}{R_{BA}}\right) > \frac{1}{\underset{\pi_k}{E^s}(R_{BA})} \tag{33}$$

so that, using Eq. (32),

$$\underset{\pi_k}{E^s}\left(\frac{1}{R_{BA}}\right) > \frac{1}{I(R_{BA})} \tag{34}$$

Comparing inequality (34) with Eq. (20) demonstrates that

$$I\left(\frac{1}{R_{BA}}\right) = \frac{1}{I(R_{BA})} < \underset{\pi_k}{E^s}\left(\frac{1}{R_{BA}}\right) \tag{35}$$

The question arises as to the value of $I(1/R_{BA})$. Noting that $R_{ABk} = V_{Ak}/V_{Bk} = 1/R_{BAk}$, inverting the valuation ratio transforms Eq. (17) into

$$I\left(\frac{1}{R_{BA}}\right) = \frac{\sum_{k=1}^{n} V_{Ak}}{\sum_{k=1}^{n} V_{Bk}} \tag{36}$$

Substituting for V_{Ak} and V_{Bk} from Eqs. (11) and (12) into Eq. (36) gives

$$I\left(\frac{1}{R_{BA}}\right) = \frac{\sum_{m=1}^{n} \min(1, R_{ABm})}{\sum_{m=1}^{n} \min\left(1, R_{ABm}^{-1}\right)} \tag{37}$$

which may be expanded to yield:

$$I\left(\frac{1}{R_{BA}}\right) = \sum_{m=1}^{n} \frac{\min\left(1, R_{ABm}^{-1}\right)}{\sum_{m=1}^{n} \min\left(1, R_{ABm}^{-1}\right)} R_{ABm} = \sum_{m=1}^{n} \rho_m R_{ABm}$$
$$= \sum_{m=1}^{n} \rho_m \frac{1}{R_{BAm}} \tag{38}$$

where ρ_k is the probability distribution:

$$\rho_k = \frac{\min\left(1, R_{ABk}^{-1}\right)}{\sum_{m=1}^{n} \min\left(1, R_{ABm}^{-1}\right)} \tag{39}$$

It is clear that the probability distribution given in Eq. (39) must be different from the probability distribution given by Eq. (24), so that

$$\rho_k \neq \pi_k \tag{40}$$

As an example, Fig. 6 shows the differing probability distributions for the data given in the numerical example at the end of Section 3. This paints a vivid picture of the shifting emphasis given to the opinion of the various respondents, depending on which Valuation Index is being calculated, $I(R_{BA})$ or $I(R_{AB})$. The respondent who considers the loss of 1 life in scenario A as bad as losing 10 lives in scenario B will be accorded great influence when $I(R_{BA})$ is being calculated, but only a small degree of influence when $I(R_{AB})$ is being evaluated. Conversely, the respondent who believes that losing 40 lives in scenario A is equivalent to losing 10 lives in scenario B will find his opinion given very little weight when $I(R_{BA})$ is being worked out, but his view will be accorded the maximum influence in the calculation of $I(R_{AB})$.

Because each must lie below the sample mean, both statistics, $I(R_{BA})$ and $I(R_{AB})$, will give emphasis to lower opinions. The fact that a low value of R_{BAk} will correspond to a high value of R_{ABk} means that the importance of a view will be upgraded in one scenario and while being downgraded at the same time in the other when there is even the smallest divergence of opinion in the sample.

Thus the reciprocality of the Valuation Index comes at the price of giving different weightings to

(i) the view of the same individual depending on whether the Valuation Index is applied to the valuation ratios to give $I(R_{BA})$ or to the reciprocals of the valuation ratios to give $I(R_{BA}^{-1}) = I(R_{AB})$, and
(ii) the views of different individuals, unless (a) all the individuals regard scenario B as no more fearsome than scenario A when the Valuation Index, $I(R_{BA})$ is

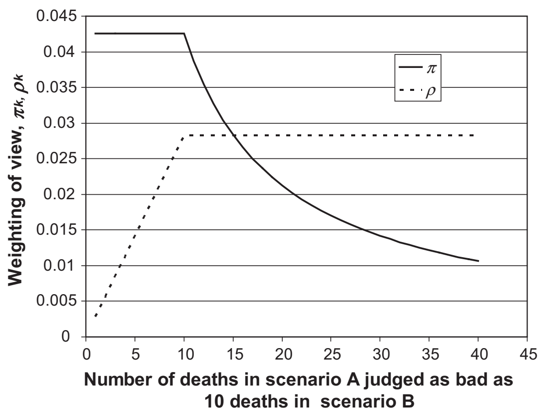

Fig. 6. Comparison of the probability distributions, π_k for $I(R_{BA})$ and ρ_k for $I(R_{AB})$.

being sought, or else (b) all the individuals regard scenario A as no more fearsome than scenario B when the Valuation Index, $I(R_{BA}^{-1}) = I(R_{AB})$ is being sought.

The only exception to (i) and (ii) is the degenerate case where all respondents respond identically.

Another perspective is given by considering the sensitivity functions associated with the formation of the Valuation Index. As noted in Section 5, and shown in Fig. 5, the change in sensitivities to the view of the same person between the formation of $I(R_{BA})$ and the formation of $I(R_{AB})$ can be enormous.

In this context, it is pertinent that the sensitivity to the view of the k^{th} person of the back-transformed mean, $Z(R_{BA})$, of a power transformation is given by

$$\frac{\partial Z(R_{BA})}{\partial R_{BAk}} = \left(\frac{R_{BAk}}{Z(R_{BA})}\right)^{b-1}\frac{1}{n} \tag{41}$$

where the back-transformed mean of the power transformation of power, b, is defined by:

$$Z(R_{BA}) = \left(\frac{1}{n}\sum_{k=1}^{n} R_{BAk}^{b}\right)^{\frac{1}{b}} \tag{42}$$

Meanwhile, for the analogously formulated, back-transformed mean of the reciprocal valuation ratios, the sensitivity to the view of the kth person is:

$$\frac{\partial Z(R_{AB})}{\partial R_{ABk}} = \left(\frac{R_{ABk}}{Z(R_{AB})}\right)^{b-1}\frac{1}{n} \tag{43}$$

The two sensitivities should be the same if the view of person k is to be treated consistently in the calculations of both the mean of the valuation ratios, $Z(R_{BA})$, and the mean of the reciprocal valuation ratios, $Z(R_{AB})$. Discounting the degenerate case where all people in the sample specify exactly the same valuation ratio, the equality of Eqs. (41) and (43) necessitates $b = 1$, rendering the back-transformed mean equal to the arithmetic or sample mean.

For the non-degenerate cases where not all the opinions are identical, the back-transformed means, $Z(R_{BA})$, of the valuation ratios, R_{BAk}, $k = 1, 2, \ldots, n$ may be classified by their relationship to the back-transformed means, $Z(R_{AB})$, of the reciprocals of the valuation ratios, R_{ABk}, $k = 1, 2, \ldots, n$, according to:

$$Z(R_{BA})\begin{cases} > \frac{1}{Z(R_{AB})} & b > 0 \\ = \frac{1}{Z(R_{AB})} & b = 0 \\ < \frac{1}{Z(R_{AB})} & b < 0 \end{cases} \tag{44}$$

The geometric mean $Z(R_{BA})|_{b=0}$ (see, e.g. [26]), possesses the property of reciprocality, but, like the Valuation Index, does so only at the cost of an inconsistent treatment of the same individual on either side of the equals sign in the central equation of inequality (44). Thus the arguments made in Thomas [26] are reinforced for the invalidity of any back-transformed mean of a power transformation, including the geometric mean, apart from a linear transformation. By contrast, the structurally independent arithmetic or sample mean emerges enhanced from the analysis and in this context as the only measure derived from a power transformation that treats individuals and their opinions consistently when calculating the back-transformed means of both the valuation ratios and the reciprocal valuation ratios.

7. The Second Valuation Index

Reciprocality is not unique to the Valuation Index. For example, there is no reason to suppose that it is not equally valid for respondent, k, to use Convention 2 to set his personal VPF. If it is assumed that all respondents will use this valuation convention, and, furthermore a ratio of means is taken, then it can be shown that the result will be the Second Valuation Index, $K(R_{BA})$, given by:

$$K(R_{BA}) = \sum_{k=1}^{n}\theta_k R_{BAk} \tag{45}$$

where

$$\theta_k = \frac{\max\left(1, R_{BAk}^{-1}\right)}{\sum_{m=1}^{n}\max\left(1, R_{BAm}^{-1}\right)} \tag{46}$$

(c.f. Eqs. (23) and (24)). This Second Valuation Index will share with the Valuation Index the property of reciprocality:

$$K\left(\frac{1}{R_{BA}}\right) = K(R_{AB}) = \frac{1}{K(R_{BA})} \tag{47}$$

where

$$K\left(R_{BA}^{-1}\right) = \sum_{k=1}^{n}\alpha_k\frac{1}{R_{BAk}} \tag{48}$$

Table 4 compares results for the sample mean, the Valuation Index and the Second Valuation Index for the numerical example of Section 5, demonstrating the reciprocality of both the Valuation Index and the Second Valuation Index.

The weightings in Eq. (48) are different from those given in Eq. (46):

$$\alpha_k = \frac{\max(1, R_{BAk})}{\sum_{m=1}^{n}\max(1, R_{BAm})} \neq \theta_k \tag{49}$$

Meanwhile, the sensitivity functions for the Second Valuation Index may be shown to be:

$$\frac{\partial K}{\partial R_{BAk}} = \frac{1}{\sum_{m=1}^{n}\max\left(1, R_{BAm}^{-1}\right)} \qquad R_{BAk} > 1 \tag{50}$$

and

$$\frac{\partial K}{\partial R_{BAk}} = \frac{\sum_{m=1}^{n} R_{BAm}\max\left(1, R_{BAm}^{-1}\right)}{R_{BAk}^{2}\left(\sum_{m=1}^{n}\max\left(1, R_{BAm}^{-1}\right)\right)^{2}} \qquad R_{BAk} \leqslant 1 \tag{51}$$

Table 4
Sample mean, Valuation Index, Second Valuation Index for the example of Section 5 in steady state.

$\overline{r_{BA}}$	1.407	$\overline{r_{AB}}$	1.203
$I(R_{BA})$	1.022	$I(R_{AB})$	0.978
$K(R_{BA})$	1.129	$K(R_{AB})$	0.886

indicating that the Second Valuation Index is structurally view dependent.

It should be emphasised that the Second Valuation Index, $K(R_{BA})$, is no more to be recommended than the Valuation Index, $I(R_{BA})$, since neither is structurally view independent. Both exhibit significantly different sensitivities to views in the sample. Similarly, the view of the same individual is given a different weighting depending on whether the Second Valuation Index is applied to the valuation ratios to give $K(R_{BA})$ or to the reciprocals of the valuation ratios to give $K\left(R_{BA}^{-1}\right) = K(R_{AB})$.

Accepting that both statistics are of questionable validity, there would appear to be no particular reason for preferring the Valuation Index to the Second Valuation Index. So if the Valuation Index were valid, then the Second Valuation Index should be equally valid. However, it is clear from Table 4 that the Valuation Index and the Second Valuation Index produce different results, implying a logical contradiction. Hence neither the Valuation Index nor the Second Valuation Index is valid.

8. Discussion

The inconsistencies and contradictions revealed above result from an attempt to find a consolidating figure with the property of reciprocality. This attempt comes up against Jensen's proof over a hundred years ago that the mean of the reciprocal will be greater than the reciprocal of the mean, not equal to it [18]. This finding will apply not only to the sample mean, where an equal weight is given to each view, but also to the Valuation Index, $I(R_{BA})$, where the weightings, $\pi_k, \sum_k \pi_k = 1$, are given to the various views. Thus the equation

$$I\left(\frac{1}{R_{BA}}\right) = I(R_{AB}) = \sum_{k=1}^{n} \rho_k R_{ABk} = \frac{1}{I(R_{BA})} = \frac{1}{\sum_{k=1}^{n} \pi_k R_{BAk}} \quad (52)$$

can work only through applying markedly different weightings, π_k, ρ_k, simultaneously to the same person's view, as demonstrated in Fig. 6. This result will apply even if no-one in the sample has a valuation ratio greater than unity, so that $I(R_{BA}) = \overline{R_{BA}}$, with the exception of the degenerate case where everyone's view is exactly the same. Such gross inconsistency in the treatment of the same person's view at the same time is the price paid for reciprocality in the Valuation Index.

The Valuation Index is not unique in possessing the property of reciprocality. The same behaviour is exhibited by the Second Valuation Index, $K(R_{BA})$, which stems from the individual measuring his higher VPF relative to his lower VPF rather than the other way round (Convention 2 rather than Convention 1 in Section 2). In this case, too, different weightings, θ_k and α_k, are needed to consolidate the views of the individuals into $K(R_{BA})$ and $K(1/R_{BA})$ so as to achieve reciprocality.

It needs to be recognised that, while an individual's view will exhibit reciprocality, the non-monolithic nature of a democratic society, where different people are allowed to have different views, means that society's view will not and, indeed, should not possess the same property. Reciprocality at a societal level would imply that everyone held exactly the same opinion, a rather disturbing notion akin to Arrow's dictator, normally precluded by assumption [1,2].

In addition to the inconsistencies just discussed, both the Valuation Index and the Second Valuation Index demonstrate structural view dependence. Discrimination against some of the views in the survey is built into the way in which these indices work. This result is of fundamental importance. It means that these indices have no place in the interpretation of opinion surveys.

For example, the Valuation Index incorporates systematic bias against the view of anyone who fears the hazard under investigation more than the base hazard. Thus if the research question is whether people fear death in multiple fatality accidents more than in the base case of a single fatality accident, the Valuation Index will downgrade the views of those with a greater fear of death in multi-fatality accidents; the greater a person's fear of death in a multiple fatality accident, the more his view will be downgraded.

Use of the Valuation Index when anyone fears the hazard under investigation more that the base hazard will result in a recommendation to spend less money on protection against the hazard in question than would be recommended using the unbiased sample mean. Thus in the focus group introduced in Section 5, suppose that four people converge on valuation ratios of 2.8, the figure applied in the UK railway industry for multiple fatality accidents from 1988 to 2003, while the remaining three respondents maintain a neutral valuation ratio of 1.0. The recommendation coming from the unbiased sample mean is for a 103% increase in the spend to protect against the multi-fatality as opposed to the single fatality accident. By contrast, the low-biased Valuation Index will call for a much lower up-rating: just 58%.

9. Conclusions

The consolidation of respondents' valuations of a non-market good, the readings, into a single figure constitutes a significant task in soft measurement. It has been shown elsewhere that when the consolidation method applied to the different views in the sample can be represented as a sequential process of transformation, averaging and back-transformation, of all the possible increasing and differentiable transformations, only linear functions preserve the necessary impartiality. This implies that the sample mean is an unbiased statistic, impartial in its treatment of the views of all respondents. But no statistic that relies on a nonlinear, increasing and differentiable transformation for its formation (the geometric mean, for example) can share this impartiality. With their minimum and maximum seeking functions, the Valuation Index and the Second Valuation Index are complicated measures that do not fit neatly into the sequential process just alluded to. Hence it has been necessary to perform the investigations detailed in this paper to find whether or not they conform to the requirements of structural view independence, and, of course, they do not. Both the Valuation Index and the

Second Valuation Index fail the test of structural view independence, meaning that each one has an inherent bias against certain views and in favour of others built into its structure. This bias means that neither should be used in the soft measurement exercise of investigating human views by opinion survey.

The Second Valuation Index is constructed using the same arguments employed to construct the Valuation Index, except that Convention 2 is used rather than Convention 1 – the individual takes as his reference point the amount he would sanction to avert death under the scenario he fears less, rather in the way that the UK rail industry used to have a baseline VPF, which used to be up rated by a factor of 2.8 to give a special VPF for multi-fatality accidents. Since Convention 2 and Convention 1 are equally valid, it follows that if the Valuation Index is valid, then so should be the Second Valuation Index. But the latter gives a different answer, implying that the validity of the Valuation Index implies its own invalidity. This constitutes a logical contradiction, the resolution of which requires that neither the Valuation Index nor the Second Valuation Index can be regarded as valid.

It has been shown that the Valuation Index's reciprocality, a property shared by the Second Valuation Index and regarded by Valuation Index proponents as an advantage, is achieved only at the price of an inconsistent treatment of the same person's view whenever there is any difference of opinion in the sample. Jensen's Inequality explains how this inconsistency is an inevitable result of the reciprocality displayed by both these indices.

The Valuation Index will be equal to the sample mean in only two cases, in both of which the spread of opinion in the sample is strongly restricted: either no respondent values the option under investigation higher than the base option or else exactly the same valuation ratio is shared by all respondents. In all other cases the Valuation Index will return a value less than the sample mean. The bias towards low valuations implies that the Valuation Index is particularly unsuitable for investigating whether or not society wants more to be spent to prevent deaths in a multiple fatality rail accident. Safety cases or regulations making use of the results of a study using the Valuation Index must be regarded as unsafe to the extent that they rely on such a study for support.

As a general point, the requirement for impartiality is so fundamental and the fact that no consolidation statistic apart from the sample mean has so far been found to be structurally view independent suggests that before a new consolidation statistic is used, it needs first to be demonstrated that it passes the test of structural view independence.

The finding has particular importance for the UK because opinion surveys interpreted using the Valuation Index have been influential in the changes to safety policy brought in for UK rail transport, where the amount recommended by the UK Rail Safety and Standards Board since 2003 to avert each death in a multiple fatality accident has been reduced to about a third of what it used to be before that date.

Appendix A. The relative size of the Valuation Index, $I(R_{BA})$ and the mean of the valuation ratio, $\overline{R_{BA}}$

It will be proved in this appendix that the Valuation Index is less than the sample mean in all except for two situations, when it will equal the sample mean.

We note first that the weighting factors, π_k, introduced in Eq. (23) satisfy the Kolmogorov conditions for a probability distribution, since $\pi_k \geqslant 0$ for all k, while summing all π_k over k gives unity:

$$\sum_{k=1}^{n}\pi_k = \sum_{k=1}^{n}\frac{\min\left(1, R_{BAk}^{-1}\right)}{\sum_{k=1}^{n}\min\left(1, R_{BAk}^{-1}\right)} = \frac{\sum_{k=1}^{n}\min\left(1, R_{BAk}^{-1}\right)}{\sum_{k=1}^{n}\min\left(1, R_{BAk}^{-1}\right)} = 1 \tag{A.1}$$

This fact makes it convenient to regard the views of the people in the sample as the population with which we will be concerned. The Valuation Index, $I(R_{BA})$, may now be seen to be the expectation of the valuation ratio, R_{BA}, for the sample population when it is assumed that the valuation ratio, R_{BAk}, of the kth person in the sample is chosen with probability, π_k:

$$I(R_{BA}) = \underset{\pi_k}{E}^{s}(R_{BA}) = \sum_{k=1}^{n}\pi_k R_{BAk} \tag{A.2}$$

Here the expectation operator, E, has been labelled with the discrete probability distribution across the sample, π_k, $k = 1, 2, \ldots, n$, and given the superscript, s, to signify that the population of interest is the sample population of n people.

Now let us assume that n_1 of the respondents set their valuation ratio less than or equal to 1.0, while n_2 set their valuation ratio greater than 1.0, so that:

$$n_1 + n_2 = n \tag{A.3}$$

Let us separate the sample of valuation ratios into two classes: less than or equal to unity on the one hand and greater than unity on the other. Ordering the respondents, k, so that $R_{BAk} \leqslant 1$ $k = 1, 2, \ldots, n_1$, while $R_{BAk} > 1$ for $k = n_1 + 1, n_1 + 2, \ldots, n$, we may now rewrite Eq. (A.2) as:

$$I(R_{BA}) = \frac{\sum_{k=1}^{n_1}\min\left(1, R_{BAk}^{-1}\right)R_{BAk} + \sum_{k=n_1+1}^{n}\min\left(1, R_{BAk}^{-1}\right)R_{BAk}}{\sum_{k=1}^{n_1}\min\left(1, R_{BAk}^{-1}\right) + \sum_{k=n_1+1}^{n}\min\left(1, R_{BAk}^{-1}\right)} = \frac{\sum_{k=1}^{n_1}R_{BAk} + n_2}{n_1 + \sum_{k=n_1+1}^{n}R_{BAk}^{-1}} \tag{A.4}$$

Eq. (A.4) may be re-expressed as

$$I(R_{BA}) = \frac{n_1\frac{1}{n_1}\sum_{k=1}^{n_1}R_{BAk} + n_2}{n_1 + n_2\frac{1}{n_2}\sum_{k=n_1+1}^{n}R_{BAk}^{-1}} = \frac{n_1 \underset{low}{E}^{s}(R_{BA}) + n_2}{n_1 + n_2 \underset{high}{E}^{s}\left(R_{BA}^{-1}\right)} \tag{A.5}$$

or

$$I(R_{BA}) = \frac{\frac{n_1}{n_2}\underset{low}{E}^s(R_{BA}) + 1}{\frac{n_1}{n_2} + \underset{high}{E}^s\left(R_{BA}^{-1}\right)} \tag{A.6}$$

where $\underset{low}{E}^s(R_{BA}) = 1/n_1\left(\sum_{k=1}^{n_1} R_{BAk}\right)$ is the expectation of the valuation ratio over the n_1 respondents in the sample whose valuation ratios were low, *viz.* less than or equal to 1.0, when each such person is chosen at random. We may note that $\underset{low}{E}^s(R_{BA})$ is identical with the sub-sample average: $\overline{R_{BA}}_{low} = \underset{low}{E}^s(R_{BA})$. In a similar fashion, $\underset{high}{E}^s\left(R_{BA}^{-1}\right) = 1/n_2\left(\sum_{k=n_1+1}^{n} R_{BA}^{-1}\right)$ is the expectation of the reciprocal of the valuation ratio over the n_2 respondents in the sample whose valuation ratios were high, *viz.* more than 1.0, when each such person is chosen at random. Once again this expectation value is identical with the subsample average: $\overline{R_{BA}^{-1}}_{high} = \underset{high}{E}^s\left(R_{BA}^{-1}\right)$.

Eq. (A.6) may be compared with the expected valuation ratio for the sample, under the assumption that each person's opinion is as likely as any other to be chosen. This may be written:

$$\begin{aligned} E^s(R_{BA}) &= \frac{1}{n}\sum_{k=1}^{n} R_{BAk} = \frac{\sum_{k=1}^{n_1} R_{BAk} + \sum_{k=n_1+1}^{n_2} R_{BAk}}{n_1 + n_2} \\ &= \frac{n_1 \underset{low}{E}^s(R_{BA}) + n_2 \underset{high}{E}^s(R_{BA})}{n_1 + n_2} \\ &= \frac{\frac{n_1}{n_2}\underset{low}{E}^s(R_{BA}) + \underset{high}{E}^s(R_{BA})}{\frac{n_1}{n_2} + 1} \end{aligned} \tag{A.7}$$

where $\underset{high}{E}^s(R_{BA}) = 1/n_2\left(\sum_{k=n_1+1}^{n} R_{BAk}\right)$. As before, this expectation value may be observed to be equal to the sub-sample mean: $\overline{R_{BA}}_{high} = \underset{high}{E}^s(R_{BA})$. Moreover, the expectation value across the whole sample is identical with the mean of the whole sample: $\overline{R_{BA}} = E^s(R_{BA})$.

We need to consider 3 cases, as follows:

Case 1: $n_2 = 0$

In this case, no-one in the sample sets his valuation ratio greater than unity, so that $R_{BAk} \leqslant 1$ for $k = 1, 2, \ldots, n$. Now $n_1 = n$ and $n_2 = 0$, and everyone is in the "low" category. Hence $\underset{low}{E}^s(R_{BA}) = E^s(R_{BA})$ and from Eq. (A.5):

$$I(R_{BA}) = \underset{low}{E}^s(R_{BA}) = E^s(R_{BA}) \tag{A.8}$$

where $E^s(R_{BA})$ is the expectation over all n respondents, equal to the whole-sample average.

Case 2: $n_1 = 0$

In this case no respondent sets his valuation ratio at unity or below, so that $n_1 = 0$ and $n_2 = n$, so that everyone is in the "high" category. Thus, from Eq. (A.5):

$$I(R_{BA}) = \frac{1}{\underset{high}{E}^s\left(R_{BA}^{-1}\right)} = \frac{1}{E^s\left(R_{BA}^{-1}\right)} \tag{A.9}$$

But Jensen's inequality [18,9] implies that

$$E(R_{BA}^{-1}) > \frac{1}{E(R_{BA})} \tag{A.10}$$

Hence

$$I(R_{BA}) < E^s(R_{BA}) \tag{A.11}$$

Thus, when everyone sets his valuation ratio above 1.0, the Valuation Index will always be less than the whole sample average.

Case 3: $n_1 > 0$ and $n_2 > 0$

In this case one or more respondents sets his valuation ratio, R_{BAk}, at or below one while others set their valuation ratios above one. Subtracting Eq. (A.6) from Eq. (A.7) gives the difference, Δ_{EI}, between the whole sample average of the valuation ratio, equal to the expected valuation ratio given by Eq. (A.7), and the Valuation Index:

$$\Delta_{EI} = \Delta\underset{low}{E} + \Delta\underset{high}{E} \tag{A.12}$$

where $\Delta\underset{low}{E}$ is defined as

$$\Delta\underset{low}{E} = \frac{n_1}{n_2}\underset{low}{E}^s(R_{BA})\left(\frac{1}{\frac{n_1}{n_2}+1} - \frac{1}{\frac{n_1}{n_2} + \underset{high}{E}^s\left(R_{BA}^{-1}\right)}\right) \tag{A.13}$$

and $\Delta\underset{high}{E}$ as

$$\Delta\underset{high}{E} = \frac{\underset{high}{E}^s(R_{BA})}{\frac{n_1}{n_2}+1} - \frac{1}{\frac{n_1}{n_2} + \underset{high}{E}^s\left(R_{BA}^{-1}\right)} \tag{A.14}$$

In the case under consideration, Case 3, where $n_2 > 0$, $\underset{high}{E}^s\left(R_{BA}^{-1}\right)$ will exist and be equal to the arithmetic average of the reciprocals of the valuation ratios where the valuation ratio is greater than unity: $R_{BAk}^{-1}|(R_{BAk} > 1)$. Hence:

$$\underset{high}{E}^s\left(R_{BA}^{-1}\right) < 1 \tag{A.15}$$

Substituting condition (A.15) into Eq. (A.13) shows that $\Delta\underset{low}{E} < 0$. Inspection of Eq. (A.13) shows that, for possible variations in $\underset{low}{E}^s(R_{BA})$, the most negative value of $\Delta\underset{low}{E}$, which we shall label $\min(\Delta\underset{low}{E})$, will occur at the limiting value, $\underset{low}{E}^s(R_{BA}) = 1$. Substituting this value of $\underset{low}{E}^s(R_{BA})$ into Eq. (A.13) gives

$$\begin{aligned} \min\left(\Delta\underset{low}{E}\right) &= \min_{\underset{low}{E^s}(R_{BA})} \Delta\underset{low}{E} \\ &= \frac{n_1}{n_2}\left(\frac{1}{\frac{n_1}{n_2}+1} - \frac{1}{\frac{n_1}{n_2} + \underset{high}{E}^s\left(R_{BA}^{-1}\right)}\right) \end{aligned} \tag{A.16}$$

The subscript, 1.0, is introduced now to signify the difference between the expected value and the Valuation Index at the point where $\underset{low}{E}^s(R_{BA}) = 1$, so that

$$\Delta_{EI_{1.0}} = \min\left(\Delta E_{low}\right) + \Delta E_{high} \quad (A.17)$$

The subscript, f, will be used to signify the difference when $E^{s}_{low}(R_{BA})$ takes a fractional value below unity: $0 \leqslant E^{s}_{low}(R_{BA}) < 1$. If the other defining parameters in equations (A.13) and (A.14), namely $E^{s}_{high}(R_{BA})$, $E^{s}_{high}(R_{BA}^{-1})$ and n_1/n_2, are kept constant then Eq. (A.16) implies that

$$\Delta_{EI_f} > \Delta_{EI_{1.0}} \quad (A.18)$$

To proceed further it is necessary to specify a probability distribution for the valuation ratios above 1.0, with the condition that the distribution must have a sharp cut-off at the lower end of the range, that is to say immediately above unity. The simplest probability distribution satisfying this requirement is the uniform distribution on the interval $(1^{+}, 2E_{high}(R_{BA}) - 1^{+})$, where 1^{+} is a number infinitesimally greater than 1.0. Under this, arguably the most general random distribution, the expectation of the reciprocal of R_{BA} may be written in terms of the expectation of R_{BA}, thus eliminating one independent variable (see Appendix C, Eq. (C.18), with K set to unity):

$$E^{s}_{high}\left(R_{BA}^{-1}\right) \approx \frac{1}{E^{s}_{high}(R_{BA})}\left(1 + \frac{1}{3}\left(\frac{2E^{s}_{high}(R_{BA}) - 2}{2E^{s}_{high}(R_{BA})}\right)^{2}\right) = \frac{1}{3\left(E^{s}_{high}(R_{BA})\right)^{3}}\left(4\left(E^{s}_{high}(R_{BA})\right)^{2} - 2E^{s}_{high}(R_{BA}) + 1\right) \quad (A.19)$$

Although diminishing in size, all successive correction terms are nevertheless positive, as demonstrated in Appendix C. Therefore the calculated value, as given by Eq. (A.19) will be less than the true value:

$$E^{s}_{high}(R_{BA}^{-1})\Big|_{c} < E^{s}_{high}\left(R_{BA}^{-1}\right) \quad (A.20)$$

Thus the magnitude of the negative terms in the two components of Δ_{EI}, as given by equations (A.13) and (A.14), will be greater when Eq. (A.19) is used as an approximation for $E^{s}_{high}\left(R_{BA}^{-1}\right)$. Hence the true difference, Δ_{EI}, will be slightly greater than the calculated difference, Δ_{EIc}:

$$\Delta_{EI} > \Delta_{EIc} \quad (A.21)$$

This situation will apply for all values of $E^{s}_{low}(R_{BA})$, including when $E^{s}_{low}(R_{BA}) = 1$, so that

$$\Delta_{EI_{1.0}} > \Delta_{EIc_{1.0}} \quad (A.22)$$

Combining condition (A.22) with that of (A.18) means that

$$\Delta_{EI_f} > \Delta_{EIc_{1.0}} \quad (A.23)$$

Thus Δ_{EI} will be greater than $\Delta_{EIc_{1.0}}$ for all possible values of $E^{s}_{low}(R_{BA})$: $0 < E^{s}_{low}(R_{BA}) \leqslant 1$. Hence we may combine the two conditions (A.22) and (A.23) into the simpler, general condition:

$$\Delta_{EI} > \Delta_{EIc_{1.0}} \quad (A.24)$$

Thus if the calculated difference, $\Delta_{EIc_{1.0}}$, at $E^{s}_{low}(R_{BA}) = 1$, is greater than zero, then it will be known that

$$\Delta_{EI} > 0 \quad (A.25)$$

for the same values of the parameters, n_1/n_2 and $E^{s}_{high}(R_{BA})$. Condition (A.24) may be tested by using Eq. (A.19) in the evaluation of equations (A.12), (A.13) and (A.14), at $E^{s}_{low}(R_{BA}) = 1$ and over a suitable range of n_1/n_2 and $E^{s}_{high}(R_{BA})$.

The results of this test are given in Fig. 7, which shows the calculated difference over the following ranges in n_1/n_2 and $E^{s}_{high}(R_{BA})$:

$$0 \leqslant \frac{n_1}{n_2} \leqslant 10$$
$$1 < E^{s}_{high}(R_{BA}) \leqslant 10 \quad (A.26)$$

The difference is always positive, apart from the limiting conditions where Case 3 degenerates into Case 1.

Fig. 8 shows $\Delta_{EI_{0.2}}, \Delta_{EI_{1.0}}$ and $\Delta_{EIc_{1.0}}$ against n_1/n_2 at two values of $E^{s}_{high}(R_{BA})$: $E^{s}_{high}(R_{BA}) = 6$ and $E^{s}_{high}(R_{BA}) = 10$. The "exact" differences, $\Delta_{EI_{0.2}}, \Delta_{EI_{1.0}}$, were found using the first thousand terms ($K = 1000$) in the convergent approximation of equation (C.15), as opposed to a single term ($K = 1$) for the "calculated difference". It is clear from Fig. 8 that $\Delta_{EI_{0.2}} > \Delta_{EI_{1.0}} > \Delta_{EIc_{1.0}}$ for both values of $E^{s}_{high}(R_{BA})$, $E^{s}_{high}(R_{BA}) = 6$ and $E^{s}_{high}(R_{BA}) = 10$.

Fig. 7 shows that the difference, Δ_{EIc}, and hence Δ_{EI}, is increasing in the expected valuation ratio, $E^{s}_{high}(R_{BA})$, amongst those who fear Scenario B more, irrespective of the ratio, n_1/n_2. The difference, Δ_{EIc}, approaches zero for the limiting condition, $E^{s}_{high}(R_{BA}) \to 1$. Given that none of the n_2 respondents can have set their valuation ratio at less

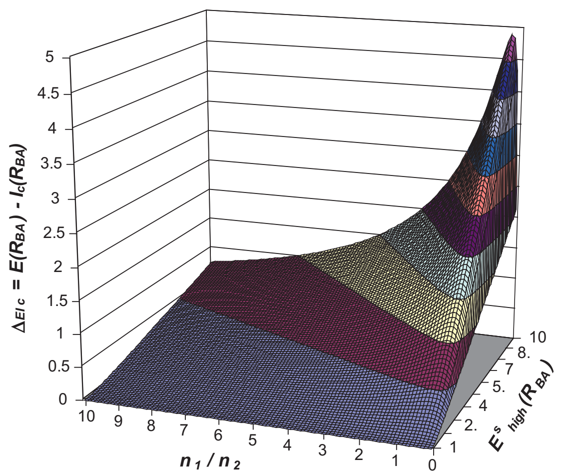

Fig. 7. The difference between the expected value, $E(R_{BA})$ and the calculated Valuation Index, $I_c(R_{BA})$. A positive difference implies that $I(R_{BA}) < E(R_{BA})$.

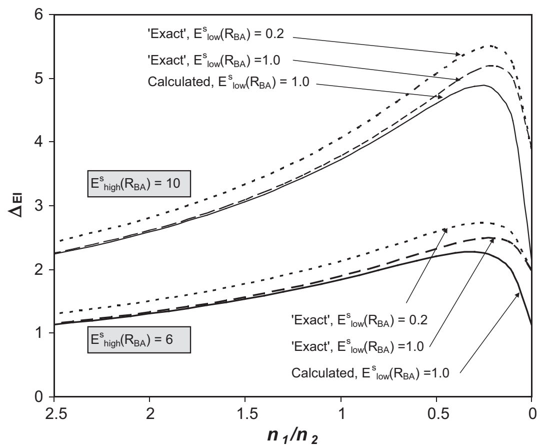

Fig. 8. The difference between the sample mean and the Valuation Index plotted against n_1/n_2. 'Exact' differences at $\underset{low}{E^s}(R_{BA}) = 0.2$ and 1.0 found using a 1000-term approximation to $\underset{high}{E^s}\left(R_{BA}^{-1}\right)$. 'Calculated' difference at $\underset{low}{E^s}(R_{BA}) = 1.0$ found from a 1-term approximation. $\underset{high}{E^s}(R_{BA}) = 10$ for the top group in the figure; $\underset{high}{E^s}(R_{BA}) = 6$ for the lower group.

than one by definition, this condition implies that all n_2 must have set their valuation ratios at 1^+. This limiting condition is consistent with Case 1, where all $n = n_1 + n_2$ respondents set their valuation at or less than unity, when the Valuation Index and the expected valuation ratio are the same, implying the difference of zero observed.

Meanwhile, a high value of n_1/n_2 implies that the large majority of the respondents belong to the group of n_1 who set their valuation ratio less than or equal to unity. The figure suggests that $\Delta_{EIc} \to 0$ as $n_1/n_2 \to \infty$, implying a smooth transition to the finding for Case 1, when $n_2 = 0$, where the Valuation Index has been shown to be equal to the expected value of the valuation ratio.

In the other limiting case where $n_1/n_2 = 0$, everyone will be in the "high" category, corresponding to Case 2. Here, from equations (A.7) and (A.9) and condition (A.10):

$$\Delta_{EI} = \underset{high}{E}{}^{s}(R_{BA}) - \frac{1}{\underset{high}{E}{}^{s}\left(R_{BA}^{-1}\right)} > 0 \tag{A.27}$$

The graph shows that there is a strongly rising trend of Δ_{EIc} with $\underset{high}{E}{}^{s}(R_{BA})$ at low values of n_1/n_2. The maximum that occurs just above $n_1/n_2 = 0$ has been shown to be slightly less than the true maximum (Fig. 8), and the fall-off towards $n_1/n_2 = 0$ is rather greater than is observed with the 1000-term calculation. The equation of the line of Δ_{EIc} with $\underset{high}{E}{}^{s}(R_{BA})$ is derived in Appendix D, and shown to be asymptotically linear in $\underset{high}{E}{}^{s}(R_{BA})$, and set to continue indefinitely.

Fig. 7 shows also that the difference between the calculated Valuation Index (and hence the Valuation Index) and the expected valuation ratio can be large – a significant fraction of the Valuation Index. The difference is large when there are many, n_2, respondents who set their valuation ratio greater than unity (so that n_1/n_2 will be low) and they set their valuation ratios high (so that $\underset{high}{E}{}^{s}(R_{BA})$ is high).

Under the mild restriction of a uniform distribution being assumed for the valuation ratios of the n_2 respondents fearing scenario B more than scenario A, it has been proved that the Valuation Index will always underestimate the expected valuation ratio where some respondents fear scenario B more than scenario A, *viz.* $n_2 > 0$, for the ranges for n_1/n_2 and $\underset{high}{E}{}^{s}(R_{BA})$ set out in condition (A.26). Put the other way round, if the valuation ratios of the n_2 correspondents fearing Scenario B more than Scenario A are uniformly distributed, *a priori* an entirely reasonable proposition, then the Valuation Index will always be less than the expected valuation ratio.

The difference between the expected valuation ratio and the Valuation Index has been observed to behave in a smooth and predictable manner when $n_2 > 0$. It is presumed, therefore, that the same result will hold if the upper limits for n_1/n_2 and $\underset{high}{E}{}^{s}(R_{BA})$ are increased without limit. The result has, in fact, been confirmed for the extended ranges, $0 \leqslant \frac{n_1}{n_2} \leqslant 25$ and $1 < \underset{high}{E^s}(R_{BA}) \leqslant 100$, which is likely to be sufficient for all practical purposes.

Considering all 3 Cases, the Valuation Index will be less than the sample mean in all except two situations, when it will equal the sample mean. Those situations are:

(i) the degenerate case of no variation between views, and
(ii) the case when no respondent fears scenario B more than scenario A.

Appendix B. The sensitivity function for the Valuation Index, *I*(R_{BA})

We will show in this appendix how expressions for the sensitivity functions of the Valuation Index can be derived.

From Eq. (23):

$$I = \pi_1 R_{BA1} + \pi_2 R_{BA2} + \ldots + \pi_i R_{BAi} + \ldots + \pi_k R_{BAk} + \ldots + \pi_n R_{BAn} \tag{B.1}$$

so that differentiating with respect to R_{BAi} yields

$$\begin{aligned}\frac{\partial I}{\partial R_{BAi}} &= R_{BA1}\frac{\partial \pi_1}{\partial R_{BAi}} + R_{BA2}\frac{\partial \pi_2}{\partial R_{BAi}} + \cdots + \pi_i + R_{BAi}\frac{\partial \pi_i}{\partial R_{BAi}} \\ &\quad + \ldots + R_{BAk}\frac{\partial \pi_k}{\partial R_{BAi}} \ldots + R_{BAn}\frac{\partial \pi_n}{\partial R_{BAi}} \\ &= \pi_i + \sum_{\substack{m=1\\ m\neq i}}^{n} R_{BAm}\frac{\partial \pi_m}{\partial R_{BAi}} + R_{BAi}\frac{\partial \pi_i}{\partial R_{BAi}}\end{aligned} \tag{B.2}$$

Differentiating Eq. (24) for $i \neq k$:

$$\begin{aligned}\frac{\partial \pi_k}{\partial R_{BAi}} &= \min\left(1, R_{BAk}^{-1}\right)\frac{\partial}{\partial R_{BAi}}\frac{1}{\sum_{m=1}^{n}\min\left(1, R_{BAm}^{-1}\right)} \\ &= \min\left(1, R_{BAk}^{-1}\right) \times \frac{-1}{\left(\sum_{m=1}^{n}\min\left(1, R_{BAm}^{-1}\right)\right)^2} \\ &\quad \times \frac{\partial}{\partial R_{BAi}}\min\left(1, R_{BAi}^{-1}\right)\end{aligned} \tag{B.3}$$

When $R_{BAi} \leqslant 1, \min\left(1, R_{BAi}^{-1}\right) = 1$, and so

$$\frac{\partial \pi_k}{\partial R_{BAi}} = 0 \quad i \neq k, \quad R_{BAi} \leqslant 1 \tag{B.4}$$

But if $R_{BAi} > 1, \min(1, R_{BAi}^{-1}) = 1/R_{BAi}$, and so $\partial \min(1, R_{BAi}^{-1}) / \partial R_{BAi} = -1/R_{BAi}^2$. Hence

$$\frac{\partial \pi_k}{\partial R_{BAi}} = \frac{1}{R_{BAi}^2} \frac{\min\left(1, R_{BAk}^{-1}\right)}{\left(\sum_{m=1}^n \min\left(1, R_{BAm}^{-1}\right)\right)^2} \quad i \neq k, \quad R_{BAi} > 1 \tag{B.5}$$

Meanwhile for $i = k$

$$\frac{\partial \pi_i}{\partial R_{BAi}} = \min\left(1, R_{BAi}^{-1}\right) \frac{\partial}{\partial R_{BAi}} \frac{1}{\sum_{m=1}^n \min\left(1, R_{BAm}^{-1}\right)} + \frac{1}{\sum_{m=1}^n \min\left(1, R_{BAm}^{-1}\right)} \frac{\partial}{\partial R_{BAi}} \min\left(1, R_{BAi}^{-1}\right) \tag{B.6}$$

When $R_{BAi} \leqslant 1$, equation (B.6) reduces to (cf. equations (B.4) and (B.5)):

$$\frac{\partial \pi_i}{\partial R_{BAi}} = 0 \quad R_{BAi} \leqslant 1 \tag{B.7}$$

When $R_{BAi} > 1, \min\left(1, R_{BAi}^{-1}\right) = 1/R_{BAi}$, Eq. (B.6) becomes

$$\begin{aligned}\frac{\partial \pi_i}{\partial R_{BAi}} &= \frac{1}{R_{BAi}^3} \frac{1}{\left(\sum_{m=1}^n \min\left(1, R_{BAm}^{-1}\right)\right)^2} - \frac{1}{R_{BAi}^2} \frac{1}{\sum_{m=1}^n \min\left(1, R_{BAm}^{-1}\right)} \\ &= \frac{1}{R_{BAi}^2 \sum_{m=1}^n \min\left(1, R_{BAm}^{-1}\right)} \left(\frac{1}{R_{BAi} \sum_{m=1}^n \min(1, R_{BAm}^{-1})} - 1\right) \quad R_{BAi} > 1\end{aligned} \tag{B.8}$$

Thus when $R_{BAi} \leqslant 1$

$$\frac{\partial I}{\partial R_{BAi}} = \pi_i \quad R_{BAi} \leqslant 1 \tag{B.9}$$

We may substitute the result that $\min\left(1, R_{BAi}^{-1}\right) = 1$ when $R_{BAi} \leqslant 1$ into Eq. (24) to give:

$$\frac{\partial I}{\partial R_{BAi}} = \frac{1}{\sum_{m=1}^n \min\left(1, R_{BAm}^{-1}\right)} \quad R_{BAi} \leqslant 1 \tag{B.10}$$

Meanwhile, when $R_{BAi} > 1$, then

$$\frac{\partial I}{\partial R_{BAi}} = \pi_i + \frac{1}{R_{BAi}^2 \left(\sum_{m=1}^n \min\left(1, R_{BAm}^{-1}\right)\right)^2} \sum_{\substack{m=1 \\ m \neq i}}^{n} \min\left(1, R_{BAm}^{-1}\right) R_{BAm} + \frac{1}{R_{BAi} \sum_{m=1}^n \min\left(1, R_{BAm}^{-1}\right)} \left(\frac{1}{R_{BAi} \sum_{m=1}^n \min(1, R_{BAm}^{-1})} - 1\right) \tag{B.11}$$

or

$$\begin{aligned}\frac{\partial I}{\partial R_{BAi}} &= \frac{\min\left(1, R_{BAi}^{-1}\right)}{\sum_{m=1}^n \min\left(1, R_{BAm}^{-1}\right)} \\ &+ \frac{1}{R_{BAi}^2 \left(\sum_{m=1}^n \min(1, R_{BAm}^{-1})\right)^2} \sum_{\substack{m=1 \\ m \neq i}}^{n} \min\left(1, R_{BAm}^{-1}\right) R_{BAm} \\ &+ \frac{1}{R_{BAi} \sum_{m=1}^n \min\left(1, R_{BAm}^{-1}\right)} \left(\frac{1 - R_{BAi} \sum_{m=1}^n \min\left(1, R_{BAm}^{-1}\right)}{R_{BAi} \sum_{m=1}^n \min\left(1, R_{BAm}^{-1}\right)}\right)\end{aligned} \tag{B.12}$$

Hence

$$\begin{aligned}\frac{\partial I}{\partial R_{BAi}} &= \frac{1}{R_{BAi} \sum_{m=1}^n \min\left(1, R_{BAm}^{-1}\right)} \\ &\times \left(1 + \frac{\sum_{\substack{m=1 \\ m \neq i}}^{n} R_{BAm} \min\left(1, R_{BAm}^{-1}\right)}{R_{BAi} \sum_{m=1}^n \min\left(1, R_{BAm}^{-1}\right)} + \frac{1 - R_{BAi} \sum_{m=1}^n \min\left(1, R_{BAm}^{-1}\right)}{R_{BAi} \sum_{m=1}^n \min\left(1, R_{BAm}^{-1}\right)}\right) \\ &= \frac{1}{R_{BAi} \sum_{m=1}^n \min\left(1, R_{BAm}^{-1}\right)} \left(\frac{1 + \sum_{\substack{m=1 \\ m \neq i}}^{n} R_{BAm} \min\left(1, R_{BAm}^{-1}\right)}{R_{BAi} \sum_{m=1}^n \min\left(1, R_{BAm}^{-1}\right)}\right) \quad R_{BAi} > 1\end{aligned} \tag{B.13}$$

Hence

$$\frac{\partial I}{\partial R_{BAi}} = \frac{1 + \sum_{m=1}^n R_{BAm} \min\left(1, R_{BAm}^{-1}\right) - R_{BAi} \min\left(1, R_{BAi}^{-1}\right)}{R_{BAi}^2 \left(\sum_{m=1}^n \min\left(1, R_{BAm}^{-1}\right)\right)^2} \quad R_{BAi} > 1 \tag{B.14}$$

and, finally

$$\frac{\partial I}{\partial R_{BAi}} = \frac{\sum_{m=1}^n R_{BAm} \min\left(1, R_{BAm}^{-1}\right)}{R_{BAi}^2 \left(\sum_{m=1}^n \min(1, R_{BAm}^{-1})\right)^2} \quad R_{BAi} > 1 \tag{B.15}$$

Appendix C. The expected value of the reciprocal of a random variable

In this appendix we derive an approximation for the expected value of the reciprocal of a random variable.

Let $h(X)$ be a continuous, non-linear function of a random variable, X, that has mean, μ. We may expand about the mean using a Taylor series:

$$\begin{aligned}h(X) = h(\mu) &+ (X-\mu)\left.\frac{dh}{dX}\right|_{X=\mu} + \frac{1}{2!}(X-\mu)^2 \left.\frac{d^2h}{dX^2}\right|_{X=\mu} \\ &+ \frac{1}{3!}(X-\mu)^3 \left.\frac{d^3h}{dX^3}\right|_{X=\mu} + \frac{1}{4!}(X-\mu)^4 \left.\frac{d^4h}{dX^4}\right|_{X=\mu} \\ &+ \frac{1}{5!}(X-\mu)^5 \left.\frac{d^5h}{dX^5}\right|_{X=\mu} + \frac{1}{6!}(X-\mu)^6 \left.\frac{d^6h}{dX^6}\right|_{X=\mu} \\ &+ \ldots\end{aligned} \tag{C.1}$$

The expected value, $E(h(X))$ is given by:

$$\begin{aligned}E(h(X)) = h(\mu) &+ \left.\frac{dh}{dX}\right|_{X=\mu} (E(X) - \mu) + \frac{1}{2!} \left.\frac{d^2h}{dX^2}\right|_{X=\mu} E((X-\mu)^2) \\ &+ \frac{1}{3!} \left.\frac{d^3h}{dX^3}\right|_{X=\mu} E((X-\mu)^3) + \frac{1}{4!} \left.\frac{d^4h}{dX^4}\right|_{X=\mu} E((X-\mu)^4) \\ &+ \frac{1}{5!} \left.\frac{d^5h}{dX^5}\right|_{X=\mu} E((X-\mu)^5) + \frac{1}{6!} \left.\frac{d^6h}{dX^6}\right|_{X=\mu} E((X-\mu)^6) + \ldots\end{aligned} \tag{C.2}$$

Since $\mu = E(X)$ by definition, the second term disappears, and so:

$$\begin{aligned}E(h(X)) &= h(\mu) + \frac{1}{2!}\frac{d^2h}{dX^2}\bigg|_{X=\mu}\mu_2 + \frac{1}{3!}\frac{d^3h}{dX^3}\bigg|_{X=\mu}\mu_3 \\ &\quad + \frac{1}{4!}\frac{d^4h}{dX^4}\bigg|_{X=\mu}\mu_4 + \frac{1}{5!}\frac{d^5h}{dX^5}\bigg|_{X=\mu}\mu_5 \\ &\quad + \frac{1}{6!}\frac{d^6h}{dX^6}\bigg|_{X=\mu}\mu_6 + \ldots + \frac{1}{r!}\frac{d^rh}{dX^r}\bigg|_{X=\mu}\mu_r + \ldots \\ &= h(\mu) + \sum_{r=2}^{\infty}\frac{1}{r!}\frac{d^rh}{dX^r}\bigg|_{X=\mu}\mu_r \end{aligned} \tag{C.3}$$

where μ_r is the rth moment taken about the mean, *viz.* the rth central moment:

$$\mu_r = E((X-\mu)^r) \tag{C.4}$$

(so that, for example, $\mu_1 = 0$, as observed already, and $\mu_2 = var(X)$).

Let us now specify the function, $h(X)$, as the reciprocal of X:

$$h(X) = \frac{1}{X} \tag{C.5}$$

Successive differentiation yields:

$$\begin{aligned}\frac{dh}{dX} &= -X^{-2} \\ \frac{d^2h}{dX^2} &= 2X^{-3} \\ \frac{d^3h}{dX^3} &= -3\times 2\times X^{-4} \\ \frac{d^4h}{dX^4} &= 4\times 3\times 2\times X^{-5} \\ &\vdots \\ \frac{d^rh}{dX^r} &= (-1)^r r!\frac{1}{X^{r+1}}\end{aligned} \tag{C.6}$$

Substituting from equations (C.5) and (C.6) into equation (C.3) gives:

$$\begin{aligned}E\left(\frac{1}{X}\right) &= h(\mu) + \sum_{r=2}^{\infty}\frac{1}{r!}(-1)^r r!\frac{1}{X^{r+1}}\bigg|_{X=\mu}\mu_r \\ &= \frac{1}{\mu} + \sum_{r=2}^{\infty}(-1)^r\frac{\mu_r}{\mu^{r+1}}\end{aligned} \tag{C.7}$$

Now let the variable, X, be uniformly distributed on (a,b). The mean is then:

$$\mu = \frac{a+b}{2} \tag{C.8}$$

while the central moments are given by [29]:

$$\mu_r = \frac{(a-b)^r + (b-a)^r}{2^{r+1}(r+1)} \tag{C.9}$$

We may develop equation (C.9) as:

$$\mu_r = \frac{(a-b)(a-b)^{r-1} + (b-a)(b-a)^{r-1}}{2^{r+1}(r+1)} \tag{C.10}$$

When r is odd, then $(r-1)$ will be even, and so

$$(a-b)^{r-1} = (b-a)^{r-1} \tag{C.11}$$

which implies from equation (C.10) that all the odd central moments will be zero:

$$\mu_r = 0 \quad \text{whenever } r \text{ is odd} \tag{C.12}$$

Hence we may rewrite equation (C.7) as:

$$E\left(\frac{1}{X}\right) = \frac{1}{\mu} + \sum_{n=1}^{\infty}\frac{\mu_{2n}}{\mu^{2n+1}} \tag{C.13}$$

Using equations (C.8) and (C.9), the ratio, $\frac{\mu_{2n}}{\mu^{2n+1}}$, emerges as:

$$\begin{aligned}\frac{\mu_{2n}}{\mu^{2n+1}} &= \frac{(a-b)^{2n} + (b-a)^{2n}}{2^{2n+1}(2n+1)} \div \frac{(a+b)^{2n+1}}{2^{2n+1}} \\ &= \frac{2(b-a)^{2n}}{(2n+1)(a+b)(a+b)^{2n}} \\ &= \frac{1}{2n+1}\frac{1}{\mu}\left(\frac{b-a}{a+b}\right)^{2n}\end{aligned} \tag{C.14}$$

Substituting from equation (C.14) into equation (C.13) gives:

$$\begin{aligned}E\left(\frac{1}{X}\right) &= \frac{1}{\mu} + \frac{1}{\mu}\sum_{n=1}^{\infty}\frac{1}{2n+1}\left(\frac{b-a}{a+b}\right)^{2n} \\ &= \frac{1}{\mu}\left(1 + \sum_{n=1}^{\infty}\frac{1}{2n+1}\left(\frac{b-a}{a+b}\right)^{2n}\right)\end{aligned} \tag{C.15}$$

It will be apparent that all the correction terms are positive. Moreover, since

$$\frac{1}{2n+1} < 1 \quad \text{for all } n \geqslant 1 \tag{C.16}$$

and

$$\frac{b-a}{a+b} < 1 \tag{C.17}$$

for all $a < b$ – the requirement for a uniform distribution – then the correction terms will be successively smaller. Thus, for any finite K,

$$E\left(\frac{1}{X}\right) > \frac{1}{\mu}\left(1 + \sum_{n=1}^{K}\frac{1}{2n+1}\left(\frac{b-a}{a+b}\right)^{2n}\right) \tag{C.18}$$

The series on the right-hand side of inequality (C.18) can be shown to be convergent for all finite b for $a > 0$. Successive approximations corresponding to $K = 1, 2, 3, \ldots$ will bring the calculation ever closer to the true solution for $E(\frac{1}{X})$. The first order approximation, taking $K = 1$, is used in Appendix A.

Appendix D. The difference between the sample mean and the calculated Valuation Index, $I(R_{BA})$, for Case 3 when $n_1/n_2 \to 0$

In this appendix we show that the difference between the sample mean and the calculated Valuation Index tends towards a linear function of the sample mean for the case when most respondents have a high valuation ratio ($R_{BA} > 1$).

Substituting from Eq. (A.19) into Eq. (A.6) gives the calculated value of the Valuation Index, $I_c(R_{BA})$, as:

$$I_c(R_{BA}) = \frac{3\left(\underset{high}{E}{}^s(R_{BA})\right)^3\left(\frac{n_1}{n_2}\underset{low}{E}{}^s(R_{BA})+1\right)}{3\frac{n_1}{n_2}\left(\underset{high}{E}{}^s(R_{BA})\right)^3+4\left(\underset{high}{E}{}^s(R_{BA})\right)^2-2\underset{high}{E}{}^s(R_{BA})+1} \tag{D.1}$$

where the term, "calculated value of the Valuation Index" is used to highlight the small degree of approximation inherent in Eq. (A.19), which will return a slightly low figure for $\underset{high}{E}{}^s(R_{BA}^{-1})$ because of the omission of the higher-order terms listed in equation (C.15). Setting

$$\begin{aligned} w &= \underset{low}{E}{}^s(R_{BA}) \\ x &= \frac{n_1}{n_2} \\ y &= \underset{high}{E}{}^s(R_{BA}) \end{aligned} \tag{D.2}$$

for compactness, we may subtract Eq. (D.1) from Eq. (A.7) to give the calculated difference between the sample mean and the calculated mean as:

$$\begin{aligned} \Delta_{EIc} &= \frac{xw+y}{x+1} - \frac{3y^3(xw+1)}{3xy^3+4y^2-2y+1} \\ &= \frac{3x(y^4-y^3)+y^3-2y^2+y-wx(3y^3-4y^2+2y-1)}{(x+1)(3xy^3+4y^2-2y+1)} \end{aligned} \tag{D.3}$$

Imposing the condition, $w = \underset{low}{E}{}^s(R_{BA}) = 1$, and also setting $x = n_1/n_2 = 0$ as a limiting condition gives the equation of the line at the extreme right-hand side of Fig. 7:

$$\Delta_{EIc_{1,0}} = \frac{y^3-2y^2-y}{4y^2-2y+1} = \frac{y}{4} \times \frac{1-\frac{2}{y}-\frac{1}{y^2}}{1-\frac{1}{2y}+\frac{1}{4y^2}} \tag{D.4}$$

so that $\Delta_{EIc_{1,0}} \to \frac{y}{4} = \frac{\underset{high}{E}{}^s(R_{BA})}{4}$ as $y \to \infty$ when $x = n_1/n_2 = 0$.

Appendix E. Model of the evolution of opinions in the focus group

The Valuation Index, I, will evolve over time from its initial value, $I_0 = 1$, according to:

$$I(t) = I_0 + \int_{\tau=0}^{t} \mathbf{bu}\, d\tau \tag{E.1}$$

where $\mathbf{b}$ is the 1×7 row vector of the sensitivity functions, $\partial I/\partial R_{BAk}$:

$$\mathbf{b} = \begin{bmatrix} \frac{\partial I}{\partial R_{BA1}} & \frac{\partial I}{\partial R_{BA2}} & \cdots & \frac{\partial I}{\partial R_{BA7}} \end{bmatrix} \tag{E.2}$$

while $\mathbf{u}$ is the column vector of the time differentials of the views:

$$\mathbf{u} = \frac{d\mathbf{r}_{\mathbf{BA}}}{dt} \tag{E.3}$$

where $\mathbf{r}_{\mathbf{BA}} = [r_{BA1} \quad r_{BA2} \quad \ldots \quad r_{BA7}]^T$ is the 7×1 column vector of views.

Meanwhile, the components of $\mathbf{u}$ are given by

$$u_k = \frac{dr_{BAk}}{dt} = \frac{r_{BAkf} - r_{BAk}}{T} \tag{E.4}$$

where T is the time constant, 80 min in this case.

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